Central Elements of Effect Algebras

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Central elements of an effect algebra can be characterized by means of a weak form of distributivity and a maximality property. We give examples where both conditions are fulfilled.

Central elements and the center (the set of all central elements) play an important role in quantum structures—they represent the "classical part" of a given model. Considering the axiomatics of quantum structures, it is important to know the impact of various conditions on the size of the center. In particular, there are a lot of results of the type that a quantum structure with some properties has to be a Boolean agebra—see, e.g., Navara and Pták (1989), Müller *et al*. (1992), Pulmannová and Majerník (1992), Müller (1993), Pulmannová (1993), Pták and Pulmannová (1994), Tkadlec (1994), Dvurečenskij and Länger (1995), Tkadlec (1995), and Navara (1997). All the results mentioned above were generalized by Tkadlec (1997) by introducing two new classes of orthomodular posets. Here we present a generalization of the latter results in two direction: (1) we consider more general structures than orthomodular posets, (2) we consider a given element instead of the whole structure.

1. BASIC NOTIONS AND PROPERTIES

Let us summarize some basic notions and properties of effect algebras. For proofs and details see, e.g., Foulis and Bennett (1994), Greechie *et al*. (1995).

Definition 1.1. An *effect algebra* is an algebraic structure $(E, 0, 1, \oplus)$ such that E is a set, 0 and 1 are different elements of E, and $\oplus$ is a partial binary operation

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on E such that for every $a, b, c \in E$ the following conditions hold (the equalities mean also "if one side exists then the other side exists"):

(1) $a \oplus b = b \oplus a$ (commutativity),
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity),
(3) for every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$ (orthosupplement),
(4) $a = 0$ whenever $a \oplus 1$ is defined (zero-unit law).

For simplicity, we use the notion E for an effect algebra. A partial ordering on an effect algebra E is defined by $a \le b$ iff there is a $c \in E$ such that $b = a \oplus c$; such an element c is unique (if it exists) and is denoted by $b \ominus a$. 0 (1, resp.) is the least (the greatest, resp.) element of E with respect to this partial ordering. An *orthogonality* relation on E is defined by $a \perp b$ iff $a \oplus b$ exists (i.e., iff $a \le b'$). It can be shown that $a \oplus 0 = a$ for every $a \in E$ and that a *cancellation law* is valid: for every $a, b, c \in E$ with $a \oplus b \le a \oplus c$ we have $b \le c$.

Definition 1.2. An *orthoalgebra* is an effect algebra E such that for every $a \in E$:

(4') $a = 0$ whenever $a \oplus a$ is defined (consistency).

Assuming conditions (1)–(3) from the definition of an effect algebra, the condition (4') is stronger than condition (4), hence we can define an orthoalbegra by conditions (1)–(3) and (4'). Other characterizations of orthoalgebras among effect algebras: $a \wedge a' = 0$ for every a; the orthosupplementation is orthocomplementation; $a \oplus b$ is a minimal upper bound for orthogonal a, b.

Definition 1.3. An *orthomodular poset* is an effect algebra E such that for every $a, b, c \in E$:

(5) $(a \oplus b) \oplus c$ is defined whenever $a \oplus b$, $b \oplus c$ and $c \oplus a$ are defined (coherence).

Obviously, if $a \perp b$ and $a \vee b$ exists in an effect algebra, then $a \vee b \le a \oplus b$. The reverse inequality need not be true.

Definition 1.4. An element a of an effect algebra E is *principal*, if $b \oplus c \le a$ for every $b, c \in E$ such that $b, c \le a$ and $b \perp c$.

If a is a principal element, then $a \wedge a' = 0$ and the interval $[0, a]$ is an effect algebra with the greatest element a and the partial operation given by restriction of $\oplus$ to $[0, a]$—the orthosupplement operation is given by $b \mapsto (b \oplus a')'$. Moreover:

Theorem 1.5. *An effect algebra is an orthomodular poset iff every its element is principal.*

Our interest will be concentrated on central elements.

Definition 1.6. An element a of an effect algebra E is *central*, if

(1) a and a' are principal,
(2) for every $b \in E$ there are $b_1, b_2 \in E$ such that $b_1 \le a$, $b_2 \le a'$, and $b = b_1 \oplus b_2$.

The *center* $C(E)$ of E is the set of all central elements of E.

The center of an effect algebra E is a sub-effect algebra of E and forms a Boolean algebra. The decomposition property of central elements (condition (2) of Definition 1.6) can be formulated by the following way: $b = (b \wedge a) \oplus (b \wedge a')$.

2. RESULTS

Let us present the main result, which generalizes Theorem 4.2 from Tkadlec (1997).

Theorem 2.1. *Let E be an effect algebra. Then $a \in E$ is central iff the following conditions hold:*

(1) *a and a' are principal,*
(2) *$b = 0$ whenever $b \in E$ with $b \wedge a = b \wedge a' = 0$,*
(3) *$[0, a] \cap [0, b]$, $[0, a'] \cap [0, b]$ have maximal elements for every $b \in E$.*

Proof: The necessity of all conditions is obvious—the first is condition (1) of Definition 1.6, the second is a weaker form of condition (2) of Definition 1.6, the third follows from the fact that for a central element a and for an arbitrary $b \in E$ the infima $b \wedge a$, $b \wedge a'$ exist.

Let us prove the sufficiency, i.e., condition (2) of Definition 1.6. Let $b \in E$. Let us denote by b_1 a maximal element of $[0, a] \cap [0, b]$ and by b_2 a maximal element of $[0, a'] \cap [0, b \ominus b_1]$ and consider the element $c = (b \ominus b_1) \ominus b_2$. Let us take an arbitrary $d \le c \wedge a$. Since $d \le c \le b \ominus b_1$, we obtain that $d \oplus b_1$ exists and $d \oplus b_1 \le b$. Since $d, b_1 \le a$ and a is principal, $d \oplus b_1 \le a$. Since b_1 is a maximal element of $[0, a] \cap [0, b]$, $d \oplus b_1 = b_1$ and therefore $d = 0$, i.e., $c \wedge a = 0$. Analogously, $c \wedge a' = 0$ and, according to assumption (2), $c = 0$. Therefore $b = b_1 \oplus b_2$ and the condition (2) from Definition 1.6 are proved. □

The following examples show that no condition from Theorem 2.1 can be omitted.

Example 2.2.

(1) Let E be the real interval $[0, 1]$ with $a \oplus b = a + b$ if $a + b \leq 1$. Then the partial ordering is the standard (total) ordering of real numbers and $a' = 1 - a$ for every $a \in E$. The condition (1) of Theorem 2.1 is fulfilled only for $a \in \{0, 1\}$ while the others are fulfilled for every $a \in E$.

(2) Let E be the family of even-element subsets of a 4-element set X, $0 = \emptyset$, $1 = X$, and $a \oplus b = a \cup b$ if a and b are disjoint. The condition (2) of Theorem 2.1 is fulfilled only for $a \in \{0, 1\}$ while the others are fulfilled for every $a \in E$.

(3) Let X_1, X_2, X_3, X_4 be pairwise disjoint infinite sets, X be their union, E consists of $X_1 \cup X_2$, $X_2 \cup X_3$, $X_3 \cup X_4$, $X_4 \cup X_1$, $\emptyset$, X, and of all subsets of X with finite symmetric difference with some of these sets. Let $a \oplus b = a \cup b$ if a and b are disjoint. The condition (3) of Theorem 2.1 is fulfilled only for a finite or cofinite subsets of X while the others are fulfilled for every $a \in E$.

Due to Theorem 1.5, we can omit condition (1) of Theorem 2.1 for orthomodular posets. If we apply the above theorem for every element of a given effect algebra, we obtain the following corollary:

Corollary 2.3. *Let E be an effect algebra such that for every $a, b \in E$ the following conditions hold:*

(1) *a is principal,*
(2) *$b = 0$ whenever $b \wedge a = b \wedge a' = 0$,*
(3) *$[0, a] \cap [0, b]$ has a maximal element.*

Then E is a Boolean algebra.

Again, for orthomodular posets we may omit the first condition and obtain thus exactly the result of Tkadlec (1997).

Properties used in Theorem 2.1 might be derived from other properties. First, let us introduce some notions.

Definition 2.4. Let E be an effect algebra. A *state s on E* is a mapping $s: E \to [0, 1]$ such that:

(1) $s(1) = 1$,
(2) $s(a \oplus b) = s(a) + s(b)$ whenever $a \oplus b$ is defined.

A state s is *Jauch–Piron* if for every $a, b \in E$ with $s(a) = s(b) = 1$ there is a $c \in E$ such that $c \leq a, b$ and $s(c) = 1$.

A set S of states on E is *unital*, if for every $a \in E \setminus \{0\}$ there is a state $s \in S$ such that $s(a) = 1$.

Obviously, for every state s we have $s(0) = 0$, $s(a') = 1 - s(a)$ for every $a \in E$, $s(a) \leq s(b)$ for $a \leq b$. By a *two-valued state* we mean a state with values in $\{0, 1\}$.

The condition (2) of Theorem 2.1 is a week form of distributivity—it can be reformulated by the following way: $b \wedge (a \vee a') = (b \wedge a) \vee (b \wedge a')$ whenever $b \wedge a = b \wedge a' = 0$. It is a consequence of a large number of various properties usually stating that we have "sufficiently enough" of "sufficiently good" states.

Proposition 2.5. *Let E be an effect algebra, $a \in E$ and at least one of the following conditions hold:*

(1) *a is principal and $b \perp a$, a' for every $b \in E$ with $b \wedge a = b \wedge a' = 0$.*
(2) *There is a $k \in [\frac{1}{2}, 1]$ and a set S of states on E such that for every $b \in E \setminus \{0\}$ with $b \wedge a = b \wedge a' = 0$ the following holds:*
   (a) *there is a state $s \in S$ such that $s(b)k$ ($s(b) > k$, resp.),*
   (b) *$s(a) + s(b) < \frac{1}{2} + k$, $s(a') + s(b) < \frac{1}{2} + k$ ($s(a) + s(b) \leq \frac{1}{2} + k$, $s(a') + s(b) \leq \frac{1}{2} + k$, resp.) whenever $s \in S$ with $s(b)k$ ($s(b) > k$, resp.).*

Then the condition (2) of Theorem 2.1 is fulfilled.

Proof:

(1) Let $b \in E$ such that $b \wedge a = b \wedge a' = 0$. Then $a \oplus b$ exists and $b \perp a'$, i.e., $b \leq a$. Since a is principal, $a \oplus b \leq a = a \oplus 0$ and therefore $b = 0$.
(2) Let $b \in E \setminus \{0\}$ such that $b \wedge a = b \wedge a' = 0$ and let us seek a contradiction. There is an $s \in S$ such that $s(b) \geq k$ ($s(b) > k$, resp.) and $s(a) + s(b) < \frac{1}{2} + k$, $s(a') + s(b) < \frac{1}{2} + k$ ($\leq$ in both inequalities, resp.). Adding these two inequalities and using the equality $s(a) + s(a') = 1$ we obtain $s(b) < k$ ($s(b) \leq k$, resp.)—a contradiction. □

Let us remark that the assumption that a is principal in condition (1) of the last proposition is required in condition (1) of Theorem 2.1. A special case of (2) we obtain for S be a "unital set of two-valued Jauch–Piron states." For other concepts (e.g. for various kinds of subadditive states) see Tkadlec (1997).

The condition (3) of Theorem 2.1 is also a consequence of various properties.

Proposition 2.6. *Let E be an effect algebra, $a \in E$ and at least one of the following conditions hold:*

(1) $b \wedge a$, $b \wedge a'$ *exist for every $b \in E$ (special case: E is a lattice),*
(2) $[0, a]$, $[0, a']$ *are closed under suprema of orthogonal elements (special case: E is orthocomplete in the lattice sence) and E is an orthoalgebra.*
(3) *There is a countable unital set of states on E and every state on E is Jauch–Piron.*

Then the condition (3) of Theorem 2.1 is fulfilled.

Proof: Let $b \in E$, we will prove the existence of a maximal element of $[0, a] \cap [0, b]$ (for a' we can proceed analogously).

(1) $a \wedge b$ is the greatest (hence a maximal) element of $[0, a] \cap [0, b]$.
(2) There is a maximal set O of mutually orthogonal elements from $[0, a] \cap [0, b]$. Then $c = \bigvee O \in E$ is a maximal element of $[0, a] \cap [0, b]$. Indeed, if $d \in [0, a] \cap [0, b]$ with $d \geq c$, then $e = d \ominus c \in [0, a] \cap [0, b]$ and $e \perp c$. Due to the maximality of O, $e \in O$ and therefore $e \perp e$. Since E is an orthoalgebra, we obtain $e = 0$ and $d = c$, i.e., c is a maximal element of $[0, a] \cap [0, b]$.
(3) If $[0, a] \cap [0, b] = \{0\}$ then the maximal element of $[0, a] \cap [0, b]$ is 0. Let us suppose that $[0, a] \cap [0, b] \neq \{0\}$ and let us denote by S a countable unital set of states on E. Then the set $S_{a,b} = \{s \in S; \ s(a) = s(b) = 1\}$ is nonempty and countable. Let s_0 be a σ-convex combination (with nonzero coefficients) of all states in $S_{a,b}$. Then $s_0(a) = s_0(b) = 1$. Since the state s_0 is Jauch–Piron, there is a $c \in [0, a] \cap [0, b]$ such that $s_0(c) = 1$. It remains to prove that c is a maximal element of $[0, a] \cap [0, b]$. Indeed, if $d \in [0, a] \cap [0, b]$ with $d \geq c$ then $e = d \ominus c \in [0, a] \cap [0, b]$ and $e \perp c$. Hence $s_0(e) = 0$ and therefore there is no state $s \in S$ such that $s(e) = 1$. Due to the unitality of S, $e = 0$ and therefore $d = c$. $\square$

The condition (3) of the above Proposition is weaker than the condition "the set of states on E is a countable unital set of Jauch–Piron states":

Example 2.7. Let E be the Boolean algebra of open subsets of the real interval $[0, 1]$. Every state on E is Jauch–Piron, the set of states is uncountable (for every $x \in [0, 1]$ there is a two-valued stated carried by x: $s_x(a) = 1$ iff $x \in a$) but there is a countable unital set of states (carried by rational numbers).

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REFERENCES

Dvurečenskij, A. and Länger, H. (1995). Bell-type inequalities in orthomodular lattices I, Inequalities of order 2. *International Journal of Theoretical Physics* **34**, 995–1024.

Foulis, D. J. and Bennett, M. K. (1994). Effect algebras and unsharp quantum logics. *Foundations of Physics* **24**, 1331–1352.

Greechie, R. J., Foulis, D., and Pulmannová, S. (1995). The center of an effect algebra. *Order* **12**, 91–106.

Müller, V. (1993). Jauch–Piron states on concrete quantum logics. *International Journal of Theoretical Physics* **32**, 433–442.

Müller, V., Pták, P., and Tkadlec, J. (1992). Concrete quantum logics with covering properties. *International Journal of Theoretical Physics* **31**, 843–854.

Navara, M. (1997). On generating finite orthomodular sublattices. *Tatra Mountains Mathematical Publications* **10**, 109–117.

Navara, M. and Pták, P. (1989). Almost Boolean orthomodular posets. *Journal of Pure and Applied Algebra* **60**, 105–111.

Pták, P. and Pulmannová, S. (1994). A measure-theoretic characterization of Boolean algebras among orthomodular lattices. *Comments on Mathematics (University of Carolina)* **35**, 205–208.

Pulmannová, S. (1993). A remark on states on orthomodular lattices. *Tatra Mountains Mathematical Publication* **2**, 209–219.

Pulmannová, S. and Majerník, V. (1992). Bell inequalities on quantum logics. *Journal of Mathematical Physics* **33**, 2173–2178.

Tkadlec, J. (1994). Boolean orthoposets—Concreteness and orthocompleteness. *Mathematics Bohemica* **119**, 123–128.

Tkadlec, J. (1995). Subadditivity of states on quantum logics. *International Journal Theoretical Physics* **34**, 1767–1774.

Tkadlec, J. (1997). Conditions that force an orthomodular poset to be a Boolean algebra. *Tatra Mountains Mathematical Publications* **10**, 55–62.